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EM 88

RESEARCH REPORT No. EM-88

Turbulent Mixing Theory Applied to Radio Scattering

RICHARD A. SILVERMAN

CONTRACT NO. AF 19(122)-42

JANUARY, 1956

NEW YORK UNIVERSITY

Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. EM-88

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Richard A. Silverman

Richard A. Silverman

Morris Kline
Project Director

The research reported in this document has been made possible through support and sponsorship extended by the Air Force Cambridge Research Center, under Contract No. AF 19(122)-42. It is published for technical information only and does not necessarily represent recommendations or conclusions of the sponsoring agency.

New York, 1956

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Abstract

Obukhoff's statistical theory of turbulent mixing is proposed as a replacement for the heuristic theories of Gallet and Villars-Weisskopf, and is applied to the problem of the scattering of radio waves by refractive index fluctuations. In the case of ionospheric scattering, order-of-magnitude agreement with the observed scattered power is obtained if the refractive index fluctuations are attributed to electron density fluctuations produced by turbulent mixing in the lower edge of the E-layer. In the case of tropospheric scattering, it appears that order-of-magnitude agreement with the observed scattered power can be obtained, except during the summer months, by attributing the refractive index fluctuations to temperature fluctuations. During the summer months and at low scattering heights, humidity and its fluctuations are expected to play a prominent role. Experimental and theoretical evidence is cited in favor of perennial fractional-degree temperature fluctuations in the troposphere. Comparison of the Obukhoff, Villars-Weisskopf, and Booker-Gordon models is given.

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1. Introduction

Recently, both Gallet[1] and Villars and Weisskopf[2] have identified turbulent mixing of refractive index inhomogeneities or gradients as the fluctuation mechanism primarily responsible for the scattering of radio waves by the troposphere and ionosphere. In each of these papers, a heuristic theory of turbulent mixing is proposed. However, there is also available a statistical theory of turbulent mixing developed by Obukhoff in 1949. This paper is concerned with the application of Obukhoff's theory to the radio scattering problem.

A brief discussion of the application of Obukhoff's theory to the radio scattering problem can be found in a report by Batchelor[3], which came to the author's attention after he had completed the work reported here. The reader's attention is also drawn to the work of Krasilnikoff[4], who, as far back as 1949, had successfully applied Obukhoff's theory to the problem of phase fluctuations and angle-of-arrival fluctuations associated with the twinkling of stars, i.e., to the problem of line-of-sight propagation through a randomly inhomogeneous atmosphere.

2. Heuristic mixing theories

Gallet[1] investigates the action of turbulent mixing in the electron density gradient at the lower edge of the E-layer of the ionosphere. Examining the experimental data and calculations of Bailey et al.[5]*, he observes that the r.m.s. fluctuation in electron density $\overline{|\Delta N/N|}$ must be $\sim 10^{-2}$, in order to account for the observed scattered power, if the scale of turbulence L_o is $\sim$100

*The calculations in this reference are based on the correlation function $\exp(-r/L_o)$, as used by Booker and Gordon[6].

meters*. He then makes the plausible assumption that fluctuations in electron density at a point are due primarily to the mixing action of the large eddies of size L_o, so that

$$\overline{|\Delta N/N|} \sim \frac{L_o}{\sqrt{3}\,\bar{N}} \text{ grad } \bar{N} ,$$

where $\bar{N}$ is the average electron density and grad $\bar{N}$ its gradient. For the values of $\bar{N}$ and grad $\bar{N}$ at the lower edge of the E-layer, the value of $\overline{|\Delta N/N|}$ calculated by this formula is indeed $\sim 10^{-2}$.**

The Villars-Weisskopf theory of turbulent mixing applies something like the same heuristics to the case of small eddies, which, presumably, determine the difference in electron density at nearby points. To get the r.m.s. difference in electron density at two points a distance L apart, Villars and Weisskopf multiply the average electron density gradient by L. This leads to an L^2-dependence for the mean-square difference in electron density, which must be regarded as strange in view of the $L^{2/3}$-dependence of the mean-square velocity difference. **Thus, it seems desirable to have recourse to a statistical theory of turbulent mixing, patterned after the Kolmogoroff theory, which has been so successful in explaining the microstructure of the turbulent velocity field. As shown in Appendix I, statistical mixing theory and the Villars-Weisskopf theory lead to quite different results.**

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*In this paper, the symbol $\sim$ means equal to within an order of magnitude, admittedly a somewhat loose relation, that must be handled with care.

**In an analogous way, Gallet estimates the value of $\overline{|\Delta N/N|}$ produced by turbulent mixing in the non-adiabatic temperature gradient in the middle of the E-layer and finds $\overline{|\Delta N/N|} \sim 10^{-2}$.

3. Obukhoff's statistical mixing theory

The required statistical theory of turbulent mixing already exists, namely, the theory developed by Obukhoff[7] and elaborated by Yaglom[8]. This is not the place to review the conceptual basis of Obukhoff's theory. It is sufficient to note that it is based on just the same sort of similarity and dimensionality arguments as those used by Kolmogoroff in his analysis of the turbulent velocity field[9]. Obukhoff's result is most simply expressed in terms of the structure function

$$D_f(r) = \overline{\left[f(P) - f(P')\right]^2} \quad , \tag{1}$$

where f(P) and f(P') are the values at the points P and P' of the scalar quantity f being mixed by the turbulence (f could be electron density, temperature, etc.), and the overbar denotes time averaging. As a result of the usual Kolmogoroff assumptions of local homogeneity and local isotropy, $D_f(r)$ depends on the magnitude of the vector joining P and P', but not on its direction or initial point, provided r is appreciably smaller than the linear dimension of the largest eddy. It then follows by similarity and dimensionality considerations that the structure function has the form[7]

$$D_f(r) = B_f^2 \, r^{2/3} \quad , \tag{2}$$

provided that r is larger than the characteristic dimension of the smallest eddy, and smaller than the characteristic dimension L_o of the largest eddy,

i.e., r must correspond to eddies in the inertial range [10]*.

Thus, the form of the structure function for a scalar field quantity mixed by turbulence is precisely that found by Kolmogoroff for the vector velocity field itself. The existence of a large range of r for which (2) is valid requires that the Reynolds number of the turbulent flow be large, a condition amply fulfilled in both the troposphere and the ionosphere. Just as in the case of the velocity field, the form of the structure function (2) implies that the (three-dimensional) spectral density of the field $f(\vec{r})$ has the form [10]

$$\Phi_f(k) = C_f k^{-5/3} .$$

To calculate cross sections for the scattering of radio waves by refractive index fluctuations, we need an expression for the mean-square coefficient $\overline{|f_{\vec{K}}|^2}$ in the Fourier expansion

$$f(\vec{r}) = \frac{1}{V} \sum_k f_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}$$

where the expansion is in the scattering volume V. To relate $\overline{|f_{\vec{K}}|^2}$ to the structure constant B_f, we proceed as follows.** We observe that

$$\overline{\left(f(\vec{r} + \vec{r}') - f(\vec{r})\right)^2} = \frac{2}{V^2} \sum_k \overline{|f_{\vec{k}}|^2} \left(1 - e^{-i\vec{k}\cdot\vec{r}}\right),$$

where we have used the fact that***

*The smallest eddy size for the scalar field f is not in general the same as the smallest eddy size for the velocity field. See [7].

**This derivation is patterned after a similar derivation for the velocity field in [10], pp. 122-123.

***That this relation holds for all k in the inertial range is a consequence of the local homogeneity. See [11].

$$\overline{f_{\vec{k}}\, f_{-\vec{k}+\vec{k}'}} = \overline{|f_{\vec{k}}|^2}\, \delta_{o\vec{k}'}.$$

If we convert this sum to an integral, we obtain

$$\overline{\left(f(\vec{r}+\vec{r}') - f(\vec{r}')\right)^2} = \frac{2}{(2\pi)^3 V} \int_o^\infty 4\pi k^2\, \overline{|f_{\vec{k}}|^2}\, \left(1 - \frac{\sin kr}{kr}\right) dk$$

$$= 2\int_o^\infty \Phi_f(k)\left(1 - \frac{\sin kr}{kr}\right) dk.$$

We now assume that the Reynolds number is so high that almost the entire spectrum lies in the inertial range where the $k^{-5/3}$-law holds. This allows us to write

$$\overline{\left(f(\vec{r}+\vec{r}') - f(\vec{r}')\right)^2} \sim 2C_f \int_o^\infty k^{-5/3} \left(1 - \frac{\sin kr}{kr}\right) dk.$$

The integral is $\frac{9}{10}\, \Gamma\,(1/3) r^{2/3}$, so that finally

$$\overline{\left(f(\vec{r}+\vec{r}') - f(\vec{r}')\right)^2} = B_f^2\, r^{2/3} = 2.41\, C_f r^{2/3},$$

i.e., the desired relation between the constants is

$$C_f = B_f^2/2.41.$$

Since

$$(3) \qquad \overline{|f_{\vec{K}}|^2} = \frac{2\pi^2 V}{K^2}\, \Phi_f(K),$$

the desired expression for $\overline{|f_{\vec{K}}|^2}$ in terms of the structure constant B_f is

$$(4) \qquad \overline{|f_{\vec{K}}|^2} = \frac{2\pi^2 V}{2.41}\, B_f^2\, K^{-11/3} \sim \frac{V}{100}\, B_f^2\, L^{11/3},$$

where $L = 2\pi/K$.*

We now derive another expression for $\overline{|f_{\vec{K}}|^2}$, this time in terms of parameters that characterize the large-scale properties of the turbulent flow. Our starting point is the expression

$$\overline{f^2} = \int_0^\infty \Phi_f(k)dk.$$

We have already noted that $\Phi_f(k)$ has a $k^{-5/3}$-dependence throughout the inertial range. As is customary**, we assume that this dependence prevails over most of the spectrum, which is a good approximation at high Reynolds number. The value of C_f is then deduced by assuming a cut-off at the largest eddy size $L_o = 2\pi/k_o$. Thus

$$\overline{f^2} \sim \int_{k_o}^\infty \Phi_f(k)dk = \frac{3}{2} C_f k_o^{-2/3},$$

or

$$\Phi_f(k) \sim \frac{2}{3} \overline{f^2}\, k_o^{2/3}\, k^{-5/3}.$$

Using (3), we have finally

$$(5) \qquad \overline{|f_{\vec{K}}|^2} = \frac{4\pi^2 V}{3} \overline{f^2}\, k_o^{2/3}\, K^{-11/3} = \frac{V}{6\pi} \frac{\overline{f^2}}{L_o^{2/3}} L^{11/3}.$$

---

* If we identify the structure function $D_f(r)$ with the Villars-Weisskopf quantity Δf_L^2, then (4) is smaller by a factor of 100 than the corresponding formula (33) of [2]. A detailed discussion of the source of error in [2] is given in [11].

**See Heisenberg [12] for a similar calculation for the velocity field.

4. Ionospheric scattering

The cross section per unit volume for scattering by electron density fluctuations in the ionosphere is [2]

$$\sigma\sim = \frac{r_e^2}{V} \overline{|N_{\vec{K}}|^2} \quad \text{cm}^{-1} \quad , \tag{6}$$

where r_e is the classical radius of the electron (2.8×10^{-13} cm.), and $\overline{|N_{\vec{K}}|^2}$ is the mean-square Fourier coefficient at wave number $\vec{K}$ of the electron density field*. In terms of the wavelength λ of the incident radiation, and the scattering angle θ, we have $K = \frac{2\pi}{L} = \frac{4\pi}{\lambda} \sin\frac{\theta}{2}$. For the usual small values of θ, $K \sim \frac{2\pi\theta}{\lambda}$ and $L \sim \frac{\lambda}{\theta}$.

Substituting from equation (5), where f is taken to be the electron density N, we have

$$\sigma\sim \sim \frac{r_e^2}{6\pi} \frac{\overline{N^2}}{L_o^{2/3}} L^{11/3} .$$

In the experiment of Bailey et al. [5], L = 1.4×10^3 cm. and $\sigma\sim$ varies from about 5×10^{-17} cm.$^{-1}$ (spring midnight) to about 5×10^{-15} cm.$^{-1}$ (summer noon) [2]. To make a comparison with these values, we must deduce a plausible value of $\overline{N^2}$, since a direct determination of $\overline{N^2}$ is out of the question. If we make Gallet's assumption that $\overline{N^2}$ is primarily due to the mixing action of the large eddies, then

$$\overline{N^2} \sim \frac{1}{3} L_o^2 (\text{grad } \overline{N})^2. \tag{7}$$

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*In equation (6) it is assumed that the incident radiation is horizontally polarized. More generally, the right-hand side of (6) must be multiplied by $\sin^2\beta$, when β is the angle between the incident electric field and the direction of scattering.

Using Gallet's value of $L_o \sim 100$ meters, and values of grad N from 1 to 4 electrons cm.$^{-3}$meter^{-1}, we find

$$\sigma \sim 10^{-14} \text{ to } 10^{-13} \text{ cm.}^{-1} ,$$

which is 10-100 times higher than representative experimental values. In view of the many assumptions implicit in this simple derivation, as well as the uncertainties in the values of L_o, grad $\bar{N}$, and even in the experimental values of the cross-section*, this must be considered 'order-of-magnitude agreement'**.

If we use (4) to express the cross section in terms of the structure constant B_N, we find

$$\sigma \sim \frac{r_e^2}{100} B_N^2 L^{11/3} .$$

The values of B_N needed to give agreement with experiment are

$$B_N \sim \frac{1}{2} \text{ to } 5 \text{ electrons cm.}^{-10/3} .$$

Direct determination of B_N transcends present experimental possibilities.

Obukhoff's theory gives a $\lambda^{11/3}$-law for the frequency dependence of the scattered power, instead of the λ^5-law predicted by Villars and Weisskopf.

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* To define a cross section, Villars and Weisskopf arbitrarily choose the value 5 km. for the effective width of the scattering region.

** Obtaining values of cross sections that are larger than experimental values is certainly not as serious a drawback for a turbulent scattering theory as obtaining values that are smaller than experimental values. This point of view can be justified as follows. Calculations of scattering cross sections are based on the assumption that the entire scattering volume is undergoing turbulent motion at high Reynolds number at all times. If, on the average, only a fraction of the scattering volume is in turbulent motion at any given time, or if the Reynolds number in parts of the scattering volume is so small that the smallest eddy size exceeds the scattering wavelength, then the calculated cross sections would be correspondingly smaller.

However, available experimental data does not permit us to choose between the exponents[13].

5. Tropospheric scattering. Observed scattering cross sections.

In the troposphere, the appropriate formula for the scattering cross section per unit volume is[2]

$$\sigma = \frac{\pi^2}{\lambda^4 V} \overline{|\varepsilon_{\vec{K}}|^2},$$

where $\overline{|\varepsilon_{\vec{K}}|^2}$ is the mean-square Fourier coefficient at wave number $\vec{K}$ of the dielectric constant field $\varepsilon(\vec{r})$. Substituting from (4), we have

$$\sigma \sim \frac{\pi^2}{100\lambda^4} L^3 D_\varepsilon(L) \sim \frac{1}{10\lambda^4} B_\varepsilon^2 L^{11/3},$$

where $D_\varepsilon(r)$ is the structure function of the dielectric constant field, and B_ε is the corresponding structure constant.

In the troposphere, ε is a function of pressure, humidity, and temperature. It has already been shown[1], [2] that pressure fluctuations are an unimportant source of fluctuations in ε, so that fluctuations in ε must be due primarily to humidity and temperature fluctuations. The meteorological equation for the dielectric constant is[4]

$$\varepsilon - 1 = \frac{p}{T}\left[2.11 \times 10^{-4} + \alpha\left(\frac{10159}{T} - 0.293\right) \times 10^{-6}\right],$$

where p is the pressure in mm. of mercury, T is the temperature in degrees Kelvin, and α is the percentage ratio of the partial pressure of water vapor

to that of dry air. Neglecting the term -0.293, we find that the change in ε due to a change in temperature is

$$|\Delta \varepsilon| \sim (1 + \frac{96\alpha}{T})\frac{Ap}{T^2} \Delta T ,$$

where $A = 2.11 \times 10^{-4}$, so that values of α of about 3 percent are required to double the effective value of A. The change in ε due to a change in α is

$$\Delta \varepsilon \sim 48 \frac{Ap}{T^2} \Delta \alpha ,$$

Thus, the relation between the structure function of the dielectric constant field and the structure functions of the temperature and humidity fields is

$$D_\varepsilon(r) \sim (1 + \frac{96\alpha}{T})^2 (\frac{Ap}{T^2})^2 D_T(r) + 2300(\frac{Ap}{T^2})^2 D_\alpha(r),$$

where it is assumed that temperature fluctuations are independent of humidity fluctuations. The corresponding formula for the scattering cross section is

$$\sigma \sim \frac{1}{10\lambda^4} (\frac{Ap}{T^2})^2 \left[(1 + \frac{96\alpha}{T})^2 B_T^2 + 2300 B_\alpha^2\right] L^{11/3}. \tag{8}$$

For comparison with experiment, we have chosen the South Dartmouth, Mass. to Alpine, N.J. link, jointly operated by the Lincoln Laboratory and the Bell Telephone Co. The link has the following specifications [14]:

Frequency $f = 3670$ Mcps.

Wavelength $\lambda = 8.17$ cm.

Half-power beam angle for 28-foot paraboloidal antennas (both at transmitter and at receiver), $\alpha_a \sim 0.65^\circ$ or 0.01135 radians (azimuth), $\alpha_e \sim 0.70^\circ$ or 0.0122 radians (elevation)

Scattering angle for midpoint of scattering volume (with 0.0° antenna beam elevation), $\theta \sim 2.75^\circ$ or 0.048 radians

Height of midpoint of scattering volume $\sim$ 5000 feet

Path length = 188 miles

The observed scattered power varies from about -91 db. below free space (winter) to about -64 db. below free space (summer)*. (See [14], Fig. 24). To calculate the observed scattering cross sections two formulas are needed. The first relates the received scattered power P_{sc} to the free space power P_{fs} that would be received with the same antennas and at the same distance:

$$P_{sc}/P_{fs} = 16\sigma V/D^2. \tag{9}$$

The other is an approximation for the scattering volume V**:

$$V \sim \left(\tfrac{1}{2} D \sin \alpha\right)^3/\sin\theta \sim D^3\alpha^3/8\theta. \tag{10}$$

Using these formulas and the data given above, we calculate the observed scattering cross section and find that it varies from about 4×10^{-13} cm.$^{-1}$ (winter) to about 2×10^{-10} cm.$^{-1}$ (summer).

In another series of experiments on the South Dartmouth to Alpine link, the transmitter and receiver antenna beams were simultaneously elevated

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*These values are monthly medians of the hourly median values of the signal level.

**More accurately, $V \sim \dfrac{D^3 a_e^2 a}{8\theta}$.

in 0.2° steps. In this way, the midpoint of the scattering volume was raised from about 5000 feet to about 14000 feet. The results of these tests are shown in Table I*.

Table I

Antenna beam elevation in degrees	Midpoint height in feet	Midpoint scattering angle in radians	Received power in db. below free space	Scattering cross section per unit volume in $cm.^{-1} \times 10^{12}$.**
0.0	5000	.048	91 to 64	.375 to 190
0.0	5000	.048	79	5.95
0.2	6500	.055	80	5.4
0.4	8000	.062	83	3.05
0.6	9500	.069	89	0.855
0.8	11000	.076	94	0.30
1.0	12500	.083	96	0.205
1.2	14000	.090	98	0.14

6. Micrometeorological interpretation.

The available measurements of the microstructure of the temperature field in the troposphere are very meager; the author knows of no direct measurements of the microstructure of the humidity field. In view of this situation, we shall calculate the values of B_T needed to account for the observed scattering, neglecting the effects of humidity, by setting α and B_α equal to

*The first entry in Table I corresponds to the annual variation of the received scattered power, which, as already noted, varies from about -91 to about -64 db. below free space. The rest of the table corresponds to the beam elevation experiments, which were made when the power received at 0.0° elevation was -79 db. (The same remarks apply to Table II).

**The reader will realize that due to the large number of approximations made, it is impossible to obtain more than rough estimates of the scattering cross sections. Two significant figures are given in the tables only to prevent unnecessary round-off errors.

zero. We shall then compare the values of B_T so obtained with experimental estimates; if a larger value of B_T is required than seems reasonable, we shall see whether the deficit in scattered power can be made up by the humidity terms. At large scattering heights, where the dryness of the air precludes any important humidity effects, it should be possible to account for the observed scattered power by temperature fluctuations alone.

Setting α and B_α equal to zero, we solve (8) for the values of $B_T(Ap/T^2)$ needed to give agreement with the observed cross sections. Then we calculate appropriate values of p and T for the various scattering heights, using the following particularizations* of formulas given by Mitra [15]:

$$T = 293 - 1.524\ h,$$

$$p = 760(1 - .0052h)^{6.86},$$

where h is the height above the earth's surface in thousands of feet. Using these values of p and T, we calculate corresponding values of B_T. The results are summarized in Table II.

Table II

Antenna beam elevation in degrees	Midpoint height in feet	Temperature in degrees K.	Pressure in mm. mercury	Value of $B_T(Ap/T^2)$ needed to give agreement with experiment x 10^8	Corresponding value of B_T**
0.0	5000	285	634	1.1 to 24	.005 to .15
0.0	5000	285	634	4.2	.025
0.2	6500	283	600	5.1	.030
0.4	8000	281	568	4.8	.030
0.6	9500	279	537	3.1	.020
0.8	11000	276	507	2.2	.015
1.0	12500	274	479	2.1	.015
1.2	14000	272	452	2.0	.015

*These formulas are based on the values p = 760 mm. and T = 293°K. at the earth's surface, and the temperature lapse rate of -5°K./km.

**To nearest multiple of .005 (or.05).

Examination of Table II shows that values of the temperature structure constant B_T of about 2 to 3 x 10^{-2} are needed to explain the observed scattered power at the time of the year when the antenna beam elevation experiments were performed. Let us examine the micrometeorological evidence in favor of values of this order of magnitude.

First, there is the experiment of Krechmer[16], who, in a series of delicate measurements with low-inertia resistance thermometers, verified the Obukhoff $r^{2/3}$-law, and found $B_T \sim 3 \times 10^{-2}$*. However, it is admittedly questionable to extrapolate Krechmer's value of B_T, which was measured near the earth's surface, to tropospheric scattering heights, since, near the earth's surface, neither the turbulent flow nor the temperature regime can be regarded as typical of the troposphere as a whole. On the other hand, the agreement is striking.

Secondly, we can borrow Gallet's idea of turbulent mixing in non-adiabatic temperature gradients. The actual temperature lapse rate in the troposphere is not adiabatic; the adiabatic lapse rate is $\sim -10^{\circ}$K./km., whereas the observed lapse rate is $\sim -5^{\circ}$K/km.[15] Thus, for the purpose of inducing temperature fluctuations, the turbulence works in a lapse rate of $\sim -5^{\circ}$/km. Comparing (4) and (5), we have

$$B_T^2 \sim 5 \overline{T^2} / L_o^{2/3} .$$

The analogue of equation (7) for the ionospheric case is

$$\overline{T^2} \sim \frac{1}{3} L_o^2 (\text{grad } \overline{T})^2,$$

*Obukhoff[7] estimated $B_T \sim 2$ to 4 x 10^{-2}. Following Obukhoff, Krasilnikoff used the value 3 x 10^{-2} in his line-of-sight calculations[4].

where grad $\overline{T}$ is the non-adiabatic temperature gradient. Choosing for L_o the value of 10^4 cm. recommended by Batchelor[3], we find

$$\overline{|\Delta T|} \sim 3/10^\circ \text{ K},$$

and

$$B_T \sim 3 \times 10^{-2},$$

in complete agreement with the other estimates of B_T. This argument suggests that away from the earth's surface, B_T will have very much the same value over quite a range of heights; according to Table II, this is indeed the case in the height range 11,000 to 14,000 feet.

The presence of fractional-degree temperature fluctuations in the troposphere and stratosphere has been amply verified by Scrase.[17] In a radiosonde experiment in late June, he found that the r.m.s. temperature fluctuation was $\sim 0.85^\circ$F. in the height range 1-12 km., $\sim 0.64^\circ$F. in the range 12-21 km., and $\sim 0.29^\circ$F. in the range 21-31 km.

To account for the greatly enhanced scattered power in the summer months, we must invoke humidity, for values of B_T such as .15 (see Table II) are about five times too large. As we have seen, humidity plays a twofold role. Not only do humidity fluctuations enhance the scattered signal directly, but temperature fluctuations cause stronger dielectric constant fluctuations in moist air than in dry air. By combining the effects of humidity with enhancement of temperature fluctuations themselves, it should be possible to explain the enhancement of the scattered signal during the summer months. Until further experimental information has been acquired, there does not seem to be much more that can be said about the role of humidity fluctuations. However, simultaneous measurements with low-inertia thermometers and microwave refractometers should permit a detailed analysis of the microstructure of the humidity

field.*

7. Wavelength dependence of the scattered signal

An examination of (8) shows that according to the Obukhoff theory, the wavelength dependence of the scattering cross section per unit volume should obey a $\lambda^{-1/3}$-law. Experimental verification of this law, which differs only weakly from the λ^{0}-law of Booker and Gordon[6] will probably not be possible until a scaled experiment at widely separated frequencies has been done. Although such an experiment is under consideration[19], it has yet to be performed**. However, data already available counterindicates the λ-law of the Villars-Weisskopf mixing theory.

Bullington et al.[20] measured simultaneous transmissions at 505 mc. and 4090 mc. over a 150 nautical mile path in Newfoundland for a full year. The power received at 4090 mc. varied seasonally from about 12 to 20 db. below that received at 505 mc. Both circuits used antennas of the same size (28 foot paraboloids) so that the beamwidth at 505 mc. should be roughly 8 times that at 4090 mc. Thus, by equation (10) the scattering volume at 505 mc. is roughly 8^3 times that at 4090 mc. If the scattering cross section per unit volume obeys a $\lambda^{-1/3}$-law, then by (9), the received power should obey a $\lambda^{8/3}$-law, i.e., the power received at 4090 mc. should be about 26 db. below that received at 505 mc. If, on the other hand, the scattering cross section obeys the Villars-Weisskopf λ-law, then the received power should obey a λ^{4}-law, i.e., the power received at 4090 mc. should be about 41 db. below that received

*Of course, the refractive index field itself, as measured directly on the microwave refractometer should follow the $r^{2/3}$-law. Refractometer measurements by Birnbaum and Bussey seem to verify this law.[18]

**In the case of ionospheric scattering, scaled experiments have been carried out.[13]

at 505 mc.

Actually, these considerations are much too unsophisticated. Before the exact wavelength dependence of the scattered power can be determined we must consider (among other factors) the effects of antenna gain degradation and the effects of scattering angle variation in the scattering volume.* In any event, it is clear that both the Booker-Gordon λ^0-law and the $\lambda^{-1/3}$-law that follows from Obukhoff's statistical mixing theory are easier to reconcile with experiment than the Villars-Weisskopf λ-law.

8. Conclusions

There are many experiments that may help test this theory of radio scattering, e.g.

1. Study of the seasonal variation of tropospherically scattered signals as a function of scattering height.

2. Test of the $\lambda^{-1/3}$-law by scaled scattering experiments.

3. Investigation of the microstructure of the temperature, humidity, and refractive index fields in the troposphere; attempts first to verify the $r^{2/3}$-law, and then to measure the structure constants of the temperature and humidity fields as functions of height, season, time of day, etc.

4. If the refractive index field n obeys the $r^{2/3}$-law, then one should check whether the observed power in a scattering experiment can be calculated from values of the structure constant of the refractive index field measured in (or near) the scattering volume. The theoretical expression for the scattering cross section per unit volume is

$$\frac{4}{10\lambda^4} B_n^2 L^{11/3} ,$$

where the factor of 4 is needed to convert dielectric constant to refractive index.

Doubtless, other relevant experiments will suggest themselves to the reader.

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* Both of these effects bring the λ^n-law more in line with the experiments of Bullington et al. Because of the antenna gain degradation, the λ^n-law is multiplied by a law weaker than λ^3 when we convert the scattering cross section per unit volume to received power. Because of the scattering angle variation, the lower frequencies scatter less than if the scattering were all at the nominal midpoint scattering angle. However, the discrepancy with experiment that has to be made up is least for the $\lambda^{-1/3}$-law.

Appendix I. Comparison of the Obukhoff and Villars-Weisskopf models.

The Villars-Weisskopf heuristic theory and the Obukhoff statistical theory of turbulent mixing lead to different expressions for the structure function. In this appendix we compare the two expressions.

According to the Villars-Weisskopf theory

$$D_f^{1/2}(L) = \overline{|f(P) - f(P')|} \sim (\text{grad } \bar{f})\, L,$$

where L is the distance between the points P and P'. On the other hand, according to the Obukhoff theory

$$D_f^{1/2}(L) \sim B_f\, L^{1/3},$$

where

$$B_f^2 \sim 5\, \overline{f^2} / L_o^{2/3} \sim (\text{grad } \bar{f})^2\, L_o^{4/3},$$

if we attribute $\overline{f^2}$ primarily to the mixing action of the large eddies of size L_o. Thus, the Obukhoff expression for the structure function is

$$D_f^{1/2}(L) \sim (\text{grad } \bar{f})\, L_o^{2/3}\, L^{1/3},$$

which agrees with the Villars-Weisskopf expression only for $L \sim L_o$. For smaller values of L, the Obukhoff theory gives larger values of $D_f^{1/2}(L)$ than the Villars-Weisskopf theory, i.e., the action of the turbulence produces local gradients steeper than the prevailing average gradient by a factor $(L_o/L)^{2/3}$. Thus, the Obukhoff model can be expected to produce more radio scattering from a given refractive index gradient than the Villars-Weisskopf model.

Finally, it should be noted that all mean-square quantities that appear in this paper are variances, i.e., the random variables f(P), f(P'), etc., can be regarded as centered about their means. This is justified by the dynamically passive nature of the quantity f, which assures that the turbulent regime is negligibly influenced even by considerable changes in $\bar{f}$ over distances of order L_o.

Appendix II. Comparison of the Obukhoff and Booker-Gordon models.

By the Booker-Gordon model we mean the use of the exponential correlation function $\exp(-r/L_o)$. For this correlation function, the scattering cross section per unit volume is[6]

$$\sigma_{BG} \sim \frac{\overline{n^2}}{2\pi\lambda^4} \frac{L^4}{L_o} ,$$

where $L \sim \lambda/\theta$ and $\overline{n^2}$ is the variance of the refractive index. On the other hand, combining the general formula for the scattering cross section per unit volume

$$\sigma = \frac{\pi^2}{\lambda^4 V} \overline{|n_{\vec{K}}|^2}$$

with equation (5), we find that according to the Obukhoff theory

$$\sigma_{Ob} \sim \frac{\overline{n^2}\pi}{6\lambda^4} \frac{L^{11/3}}{L_o^{2/3}} .$$

The ratio of the Obukhoff and Booker-Gordon expressions is

$$\frac{\sigma_{Ob}}{\sigma_{BG}} \sim 3(L_o/L)^{1/3} .$$

Thus, the Obukhoff model predicts more scattering from a given variance of refractive index than the Booker-Gordon model by a factor $(L_o/L)^{1/3}$. (Compare with analogous result in Appendix I.) This fact is striking in view of recent failures to account for the observed scattering using the Booker-Gordon model.[21]

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TITLE Turbulent mixing theory applied to radio scattering.

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